Planar graphs are measure treeable

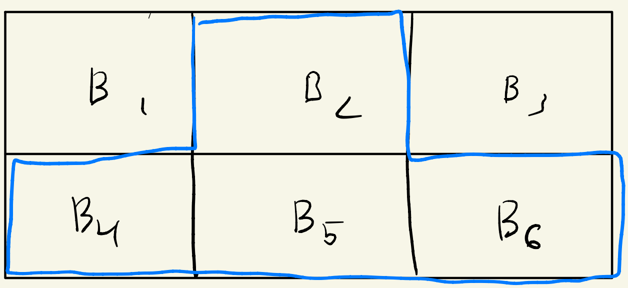
Recall

2-basis of a graph is a collection $\mathcal{B}$ of simple cycles s.th

i) no edge e belongs to more than two elts of $\mathcal{B}$

ii) $\mathcal{B}$ generates all cycles ie if C is a cycle in G, then there are $B_1, \dots, B_n \in \mathcal{B}$

$$\mathbb{1}_C = \sum_{i=1}^{n} \mathbb{1}_{B_i} \pmod 2$$



An accumulation-free planar embedding of a graph is a planar embedding s.th every compact subset of $\mathbb{R}^2$ intersects finitely many vertices and edges

A *face* of a planar embedding is a ~~n~~ bdd connected component of the complement of the embedding

If F is a face ∂F is either a cycle or a bi-infinite line

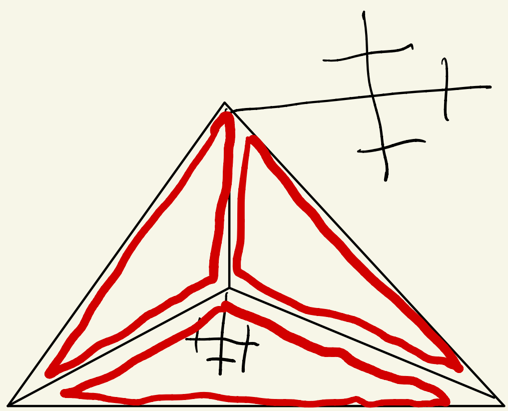
If it is a cycle, then we call it a *facial cycle*.

Thm (Thomassen) G 2-connected locally finite graph

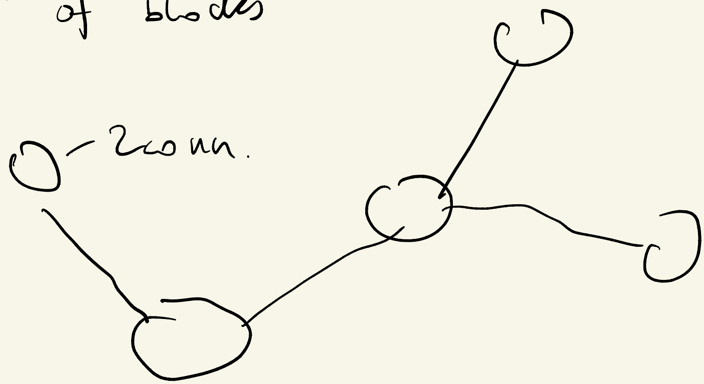
1) If G admits an accumulation-free planar embedding, then $\mathcal{B}$ the set of all facial cycles is a 2-basis of G

2) If G has a 2-basis $\mathcal{B}$, then there exists an acc. free planar embedding s.th. $\mathcal{B}$ is the set of all facial cycles.

Remark 1)



2) any graph has canonical <u>blocks</u> which are maximal 2-connected components and the graph is a "tree" of blocks

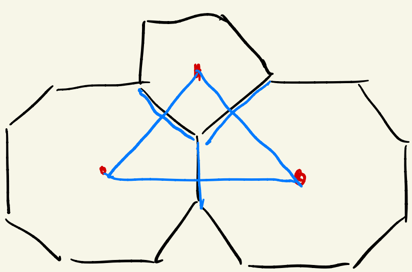


The dual of graph G with a 2-basis $\mathcal{B}$

G^* whose vertices are the elements of $\mathcal{B}$

if e is an edge in G s.th e belongs to two elts of $\mathcal{B}$, say B_1, B_2

e^* is edge between B_1, B_2



Proposition G locally finite 2-connected graph which has a 2-basis $\mathcal{B}$. Assume that every edge belongs to two elements of $\mathcal{B}$.

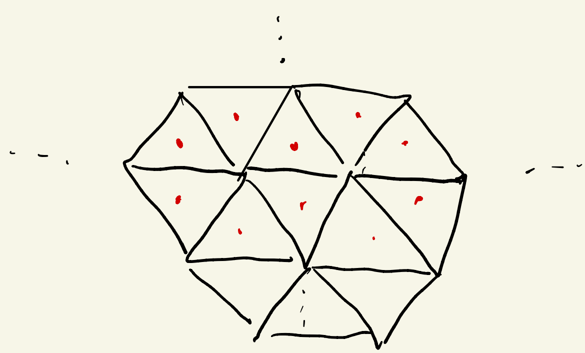
Then G^* has a 2-basis: if v is a vertex in G

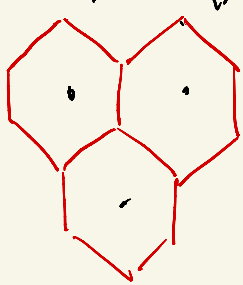
$$v^* = \{ e^* : v \in e \} \text{ is a simple cycle in } G^*$$

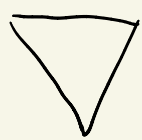
$$\mathcal{B}^* = \{ v^* : v \in G \} \text{ is a 2-basis of } G^*$$

$$G \simeq G^{**}$$

$$v \longmapsto v^*$$



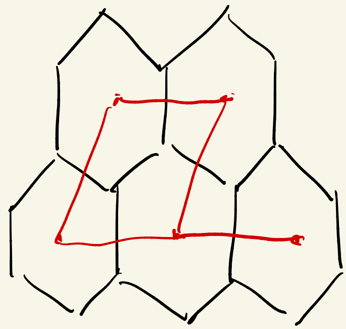


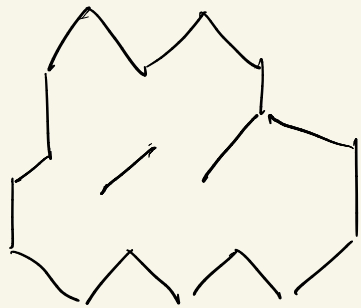


Def Suppose G is a planar graph with a 2-basis $\mathcal{B}$, G^* - its dual, $G_0^* \subseteq G^*$ is a subgraph

$$G \ominus G_0^* = G \setminus \{ e \in G : e^* \in G_0^* \}$$

Ex





Thm G locally finite aperiodic, 2-connected graph with a 2-basis $\mathcal{B}$.
Assume every edge of G belongs to two elements of $\mathcal{B}$.
G^* - dual. $G_0^* \subseteq G^*$ spanning subgraph

$$H = G \ominus G_0^*$$

1) H is acyclic iff G_0^* is aperiodic

2) G_0^* is acyclic iff H is aperiodic

3) H is a spanning tree (with the same components as G)

iff

G_0^* is a one-ended subtree.

pf Assume G, G^* are connected

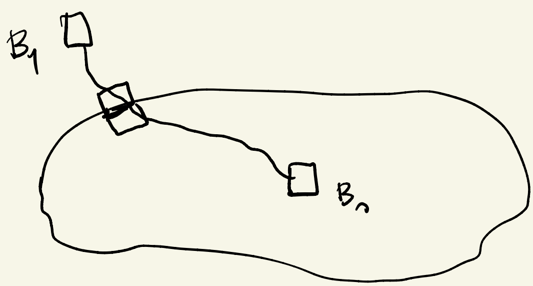
1) $\Rightarrow$

suppose G_0^* is aperiodic

Let C be a cycle in G

we want to show that C does not survive to H

we draw C on the plane and see it as a simple curve in $\mathbb{R}^2$



By Jordan H divides $\mathbb{R}^2$ into two conn. components.

one bounded, one unbounded.

Let B_0 be a facial cycle in the bounded component

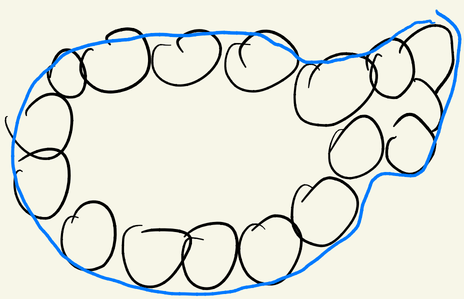
There are fin. many facial cycles in the bd connected comp., so there is a facial cycle B_1 in the unbounded comp. s.t. B_0, B_1 are connected in G_0^*

the edge cut by a path from B_0 to B_1 does not survive to H.

$\Leftarrow$ suppose G_0^* has a finite component $\mathcal{C}$

we will find a cycle in H

F



Let $K = \cup \mathcal{C}$

Let F be an unbounded face of K

∂F - is the cycle

∂F is contained in H because every edge on ∂F touches only one facial cycle of G_0^*

2) follows from 1) by duality

3)

$\Leftarrow$ assume G_0^* is a one-ended subtree

WTS H is a spanning tree (connected)

By 1) H is acyclic, so we need to show H is connected

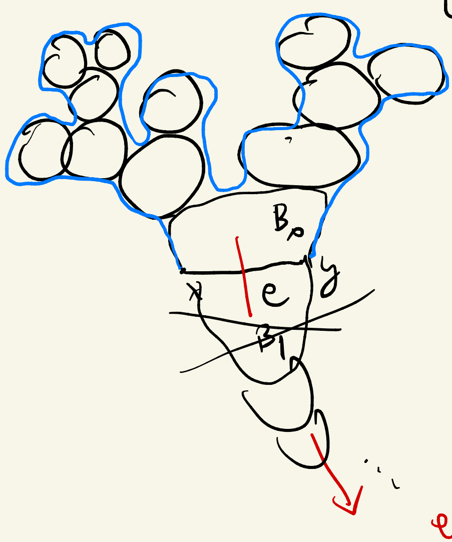
Let $x, y \in H$ assume there is an edge e between them in G.

WTS that there is a path in H from x to y.

Let B_0, B_1 be facial cycles adjacent to e, B_1 closer to the end

Removing B_1 leaves B_0 in a finite component of G_0^*

call it D



Let $P = \partial D \setminus \{e\}$ (the blue path)

every element of P survives in H,

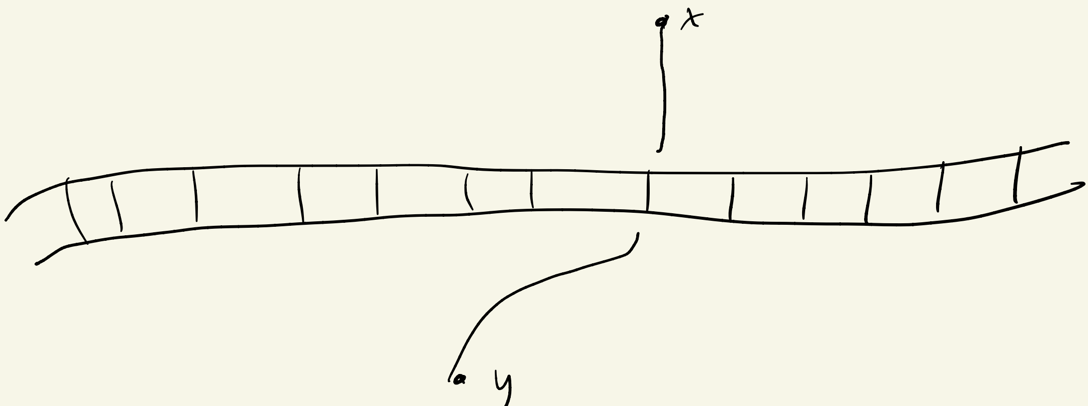
so x and y are connected in H.

$\Longrightarrow$ assume H is connected tree

WTS G_0^* is a one-ended subtree

By 2) G_0^* is a subtree, all that is left to see is that G_0^* is one-ended

If G_0^* is not one-ended, then H (it is aperiodic), it has a biinfinite path



If x, y are in different components of its complement, then x, y are not connected in H. $\square$